ITB Graduate School Conference 2021

A Note on Stefan Wave Problem

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CONTENT

Introduction

Introduction

String vibration with an obstacle on one end gives a time-dependent boundary

Modelling of Sitar music instruments arises this problem

This moving boundary problem, i.e. Stefan Problem, originates in phase-changing problem

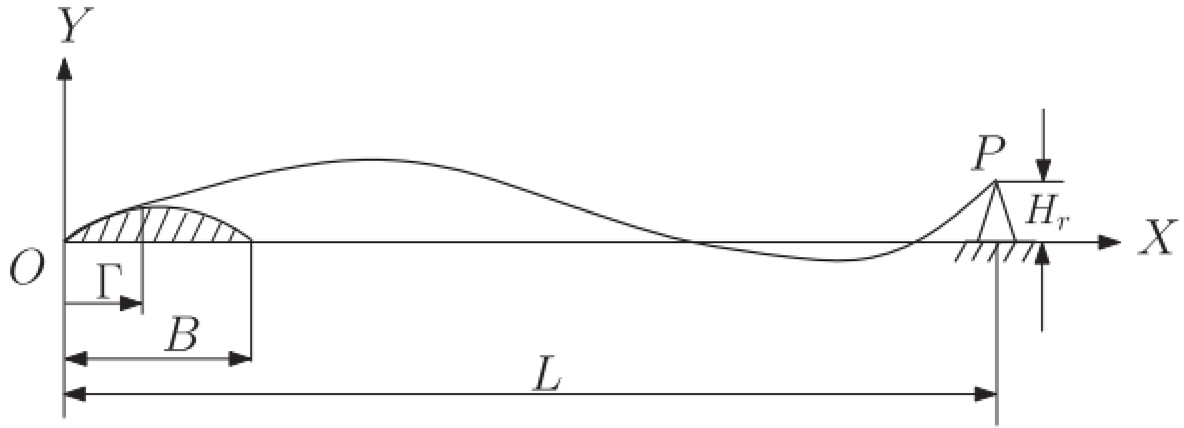
Stefan problem rarely has exact solution. Approximations are needed

In this note, perturbation and numerical approach are implemented to study the problem

Formulation

Let a string with displacements denoted by $Y = (X, \theta)$
Its vibrations is governed by

$$TY_{XX} = \rho Y_{\theta\theta}, \qquad \Gamma(\theta) \le X \le L$$
$$Y(\Gamma,\theta) = Y_B(\Gamma) = A_p\Gamma(B-\Gamma)$$
$$Y(L,\theta) = H_r$$
$$Y_X(L,\theta) = \frac{dY_B}{dX}(\Gamma) = A_p(B-2\Gamma)$$



Ilustration of the problem (Mandal and Wahi, 2015)

Non-Dimensionalization

Define scaling transformation:

$$\bar{x} = \frac{X}{L}, \qquad \bar{y}(\bar{x},\bar{\tau}) = \frac{Y(X,\theta)}{A_p L^2}, \qquad \bar{\tau} = \theta\sqrt{\frac{T}{\rho L^2}}, \qquad \bar{\gamma} = \frac{\Gamma}{L}, \qquad b = \frac{B}{L}, \qquad h = \frac{H_r}{A_p L^2}$$

The problem becomes:

$$\bar{y}_{\bar{x}\bar{x}} - \bar{y}_{\bar{\tau}\bar{\tau}} = 0, \qquad \bar{\gamma}(\bar{\tau}) \le \bar{x} \le 1$$
$$\bar{y}(\bar{\gamma}(\bar{\tau}),\bar{\tau}) = \bar{\gamma}(b - \bar{\gamma})$$
$$\bar{y}(1,\bar{\tau}) = h$$
$$\bar{y}_{\bar{x}}(1,\bar{\tau}) = b - 2\bar{\gamma}$$

Define also initial conditions:

$$\bar{y}(\bar{x},0) = \bar{f}(\bar{x}) \qquad \gamma(0) = P$$
$$\bar{y}_{\bar{\tau}}(\bar{x},0) = \bar{g}(\bar{x}) \qquad \dot{\gamma}(0) = Q$$

where P and Q satisfy: $(b - P) = \bar{f}(P)$

$$Q(b - 2P) = \bar{g}(P)$$

Boundary Fixing

Define another transformation:

$$x = \frac{\bar{x} - \bar{\gamma}}{1 - \bar{\gamma}}, \qquad y(x,\tau) = \bar{y}(\bar{x},\bar{\tau}), \qquad \tau = \tau(\bar{\tau}), \qquad \gamma(\tau) = \bar{\gamma}(\bar{\tau})$$

The problem becomes nonlinear boundary value problem

$$\begin{aligned} \left[1 - \left((x-1)\dot{\gamma}\tau_{\bar{\tau}}\right)^2\right] y_{xx} &= (1-\gamma)^2 \tau_{\bar{\tau}}^2 y_{\tau\tau} + 2(x-1)(1-\gamma)\dot{\gamma}\tau_{\bar{\tau}}^2 y_{x\tau} \\ &\quad + (x-1)[2\dot{\gamma}^2 + (1-\gamma)\ddot{\gamma}]\tau_{\bar{\tau}}^2 y_x + (1-\gamma)^2 \tau_{\bar{\tau}\bar{\tau}} y_\tau, \qquad 0 < x < 1 \\ y(0,\tau) &= \gamma(b-\gamma) \\ y(1,\tau) &= h \\ y_x(0,\tau) &= (b-2\gamma)(1-\gamma) \end{aligned}$$

Analytical Solution

Perturbation Expansion

Assume perturbation solutions:

$$y(x,\tau) = z(x) + \varepsilon v(x,\tau) + \mathcal{O}(\varepsilon^2)$$
$$\gamma(\tau) = \gamma_0 + \varepsilon s(\tau) + \mathcal{O}(\varepsilon^2)$$

Thus, we have:

$\mathcal{O}(1)$:

$$\begin{cases} z'' = 0 \\ z(0) = \gamma_0(b-\gamma_0) \\ z(1) = h \\ z'(0) = (b-2\gamma_0)(1-\gamma_0) \end{cases}$$

$\mathcal{O}(\varepsilon)$:

$$\begin{cases} (1-x)(1-\gamma_0)\ddot{s}\tau_{\bar{\bar{\tau}}}^{2}z' + v_{xx} = (1-\gamma_0)^2\tau_{\bar{\bar{\tau}}}^{2}v_{\tau\tau} + (1-\gamma_0)^2\tau_{\bar{\bar{\tau}}\bar{\bar{\tau}}}v_{\tau} \\ v(0,\tau) = s(b-2\gamma_0) \\ v(1,\tau) = 0 \\ v_x(0,\tau) = -s(b-4\gamma_0+2) \end{cases}$$

Perturbation Expansion

Solving $\mathcal{O}(1)$ problem gives us:

$$z\,(x) = hx + \gamma_0(b - \gamma_0)(1 - x)$$
$$\gamma_0 = 1 - \sqrt{1 + h - b}$$

Defining transformations

$$\bar{\tau} = \tau(1 - \gamma_0)$$
$$w(x,\tau) = v(x,\tau) - (b - 2\gamma_0)(1 - x)s(\tau)$$

Order $\mathcal{O}(\varepsilon)$ problem becomes:

$$w_{\tau\tau} = w_{xx}$$
$$w(0,\tau) = w(1,\tau) = 0$$

Perturbation Expansion

Solving $\mathcal{O}(\varepsilon)$ problem gives us:

$$w(x,\tau) = \sum_{n=}^{\infty} \phi_n(\tau)\psi_n(x)$$

$$s(\tau) = \sum_{n=1}^{\infty} \frac{\omega_n \phi_n(\tau)}{2(\gamma_0 - 1)}$$

where:

$$\phi_n(\tau) = A_n \sin(\omega_n \tau) + B_n \cos(\omega_n \tau)$$
$$\psi_n(x) = \sin(\omega_n x)$$
$$A_n = -\frac{2}{\omega_n^2}[g(0) - \omega_n \langle g, \psi \rangle]$$
$$B_n = -\frac{2}{\omega_n}[f(0) - \omega_n \langle f, \psi \rangle]$$
$$\omega_n = n\pi$$

Numerical Solution

Numerical Computation

Denote: $v_i^n = v(i\Delta x, n\Delta t)$ and $s^n = s(n\Delta t)$

Using central difference scheme:

n=0: $$v_i^0 = f(i\Delta x)$$ $$s^0 = \frac{(P-\gamma_0)}{\varepsilon}$$

n=1: $$v_i^1 = \frac{1}{2}V_i^0 + \Delta t(1-\gamma_0)g(i\Delta x) - C_i(s^1 - p - q\Delta t)$$ $$s^1 = \frac{1}{2(\gamma_0-1)\Delta x}\left[\frac{1}{2}V_i^0 + \Delta t(1-\gamma_0)g(\Delta x) + C_1(p+q\Delta t)\right]$$

Rest of time steps: $$v_i^{n+1} = V_i^n - v_i^{n-1} - C_i(s^{n+1} - 2s^n + s^{n-1})$$ $$s^{n+1} = \frac{1}{2(\gamma_0-1)\Delta x}[V_1^n - v_1^{n-1} + C_1(2s^n - s^{n-1})$$

where: $$V_i^n = 2v_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2(v_{i+1}^n - 2v_i^n + v_{i-1}^n)$$ $$C_i = (i\Delta x - 1)(b - 2\gamma_0)$$

With BC: $v_0^n = s^n(b-2\gamma_0)$ and $v_{N_x}^n = 0$

ANALYSIS AND DISCUSSION

Analysis and Discussion

Observation: value of ε needs to be very small due to the scale of variables and parameters involved.

Illustration: if for example $b = 0.1$, then the obstacle height will be $\frac{b^2}{4} = 0.0025$. The height of other fixed end should be in this scale too. Thus, the vibration should be extremely small to make this realistic.

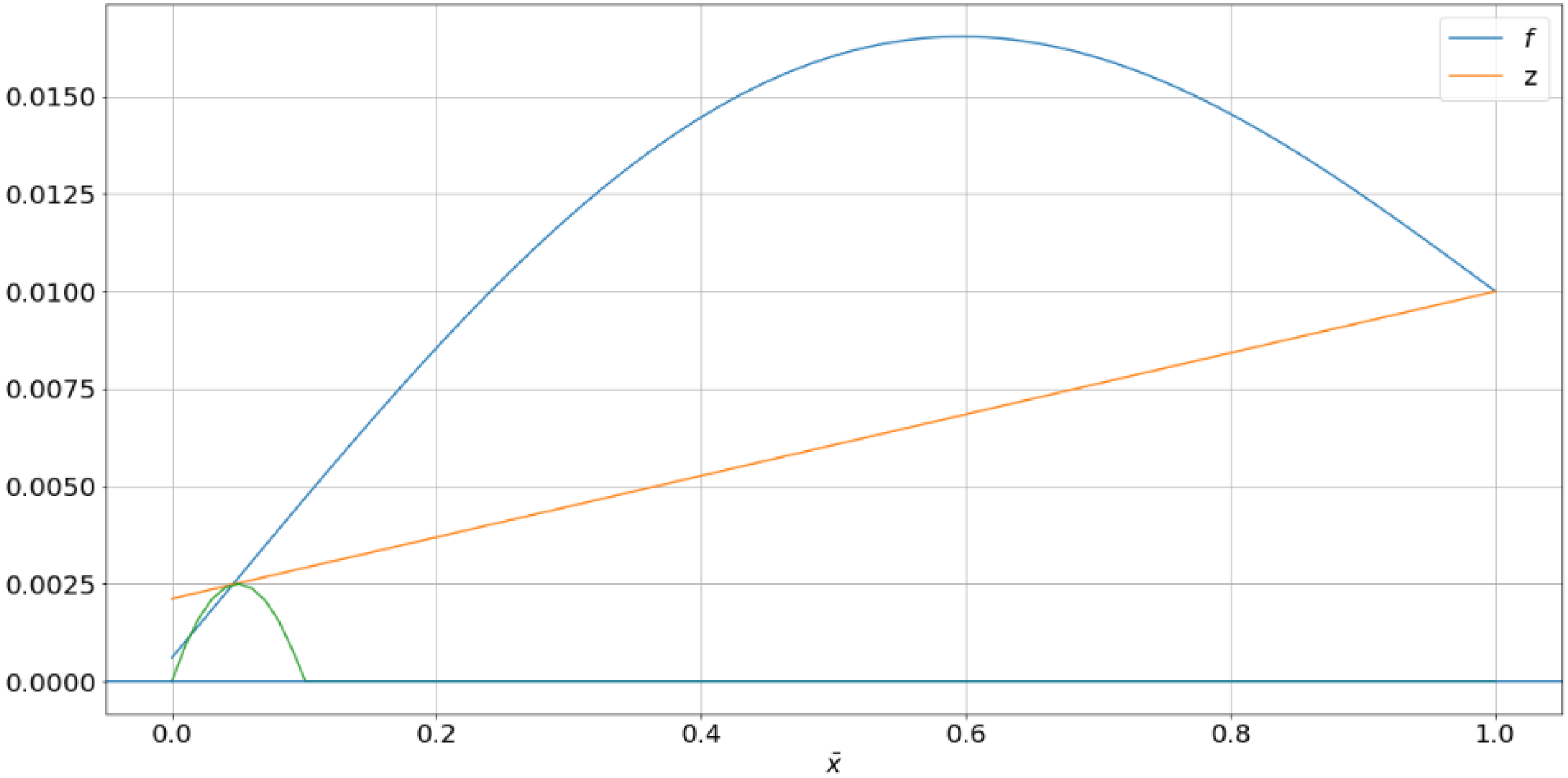
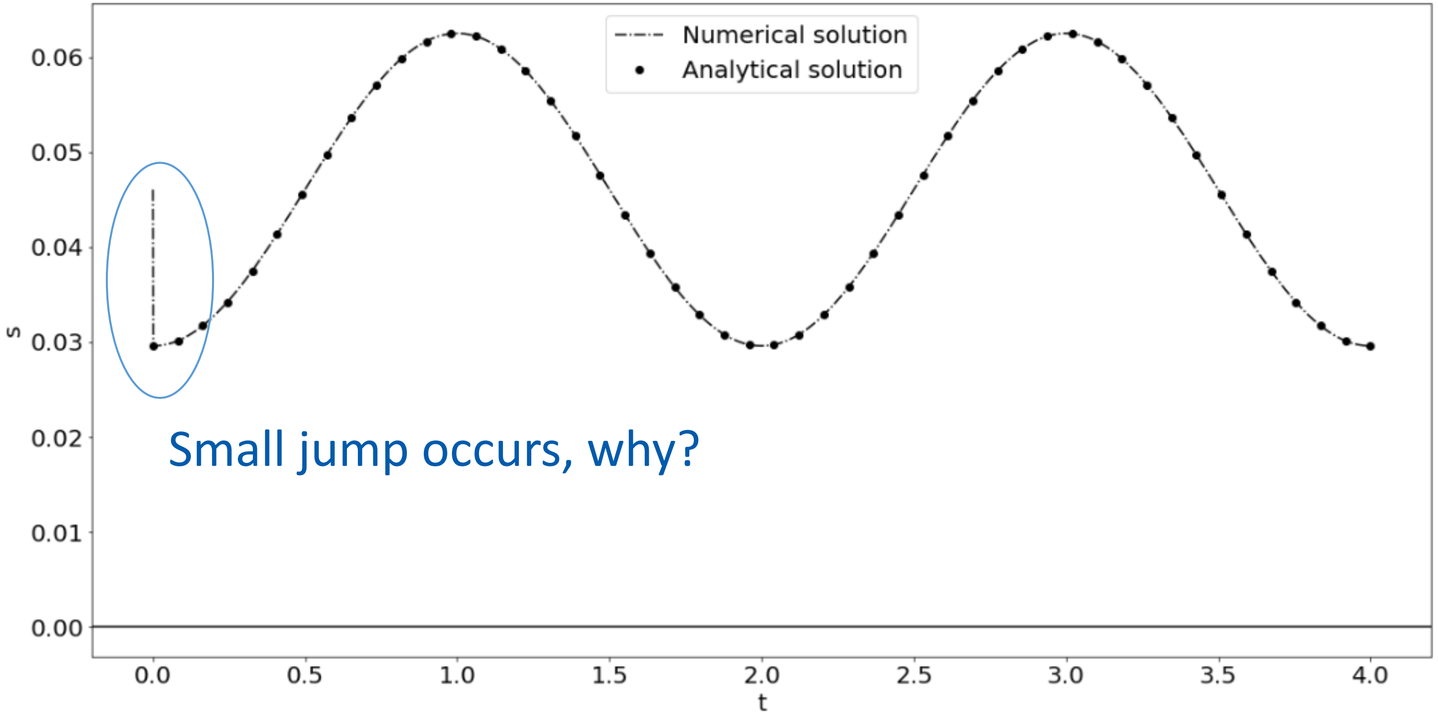


Illustration if we take $\varepsilon = h = 0.01$, and $f(x) = \sin \pi x$

Analysis and Discussion

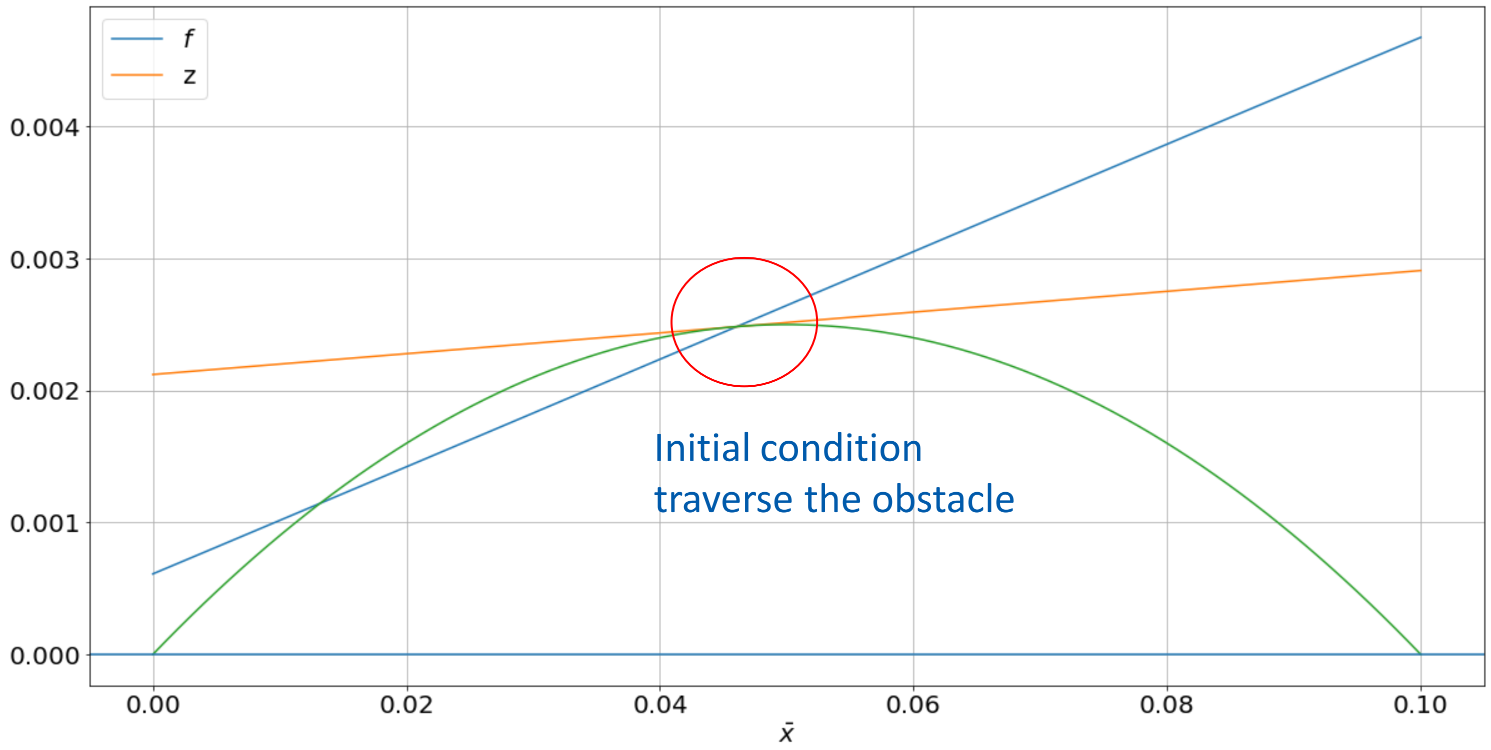
Using
$h = \varepsilon = 0.01,\ b = 0.1,$
$f(x) = \sin(\pi x)$, and
$g(x) = 0$



Analysis and Discussion

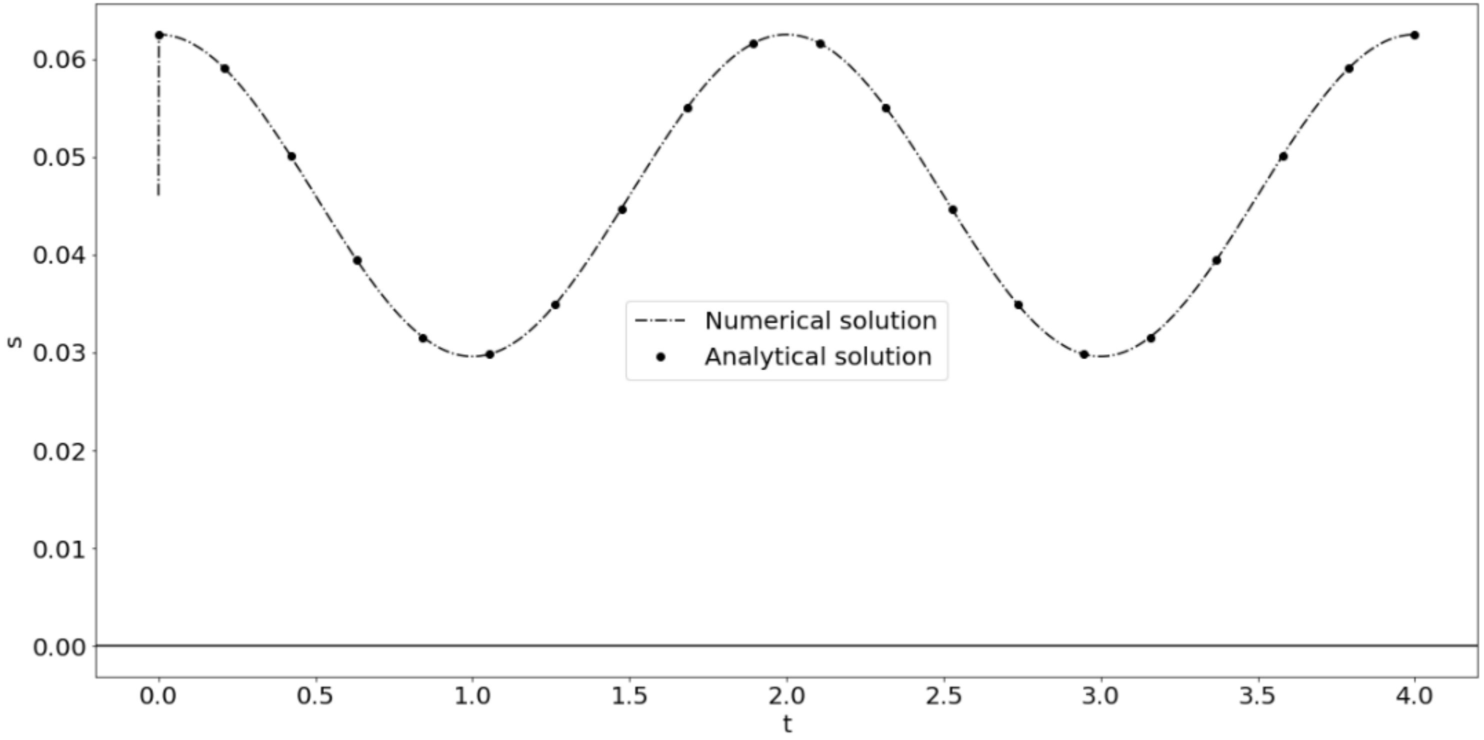
The initial condition f violates tangential condition at boundary.

So, after first step computation, the attachment point is "corrected"



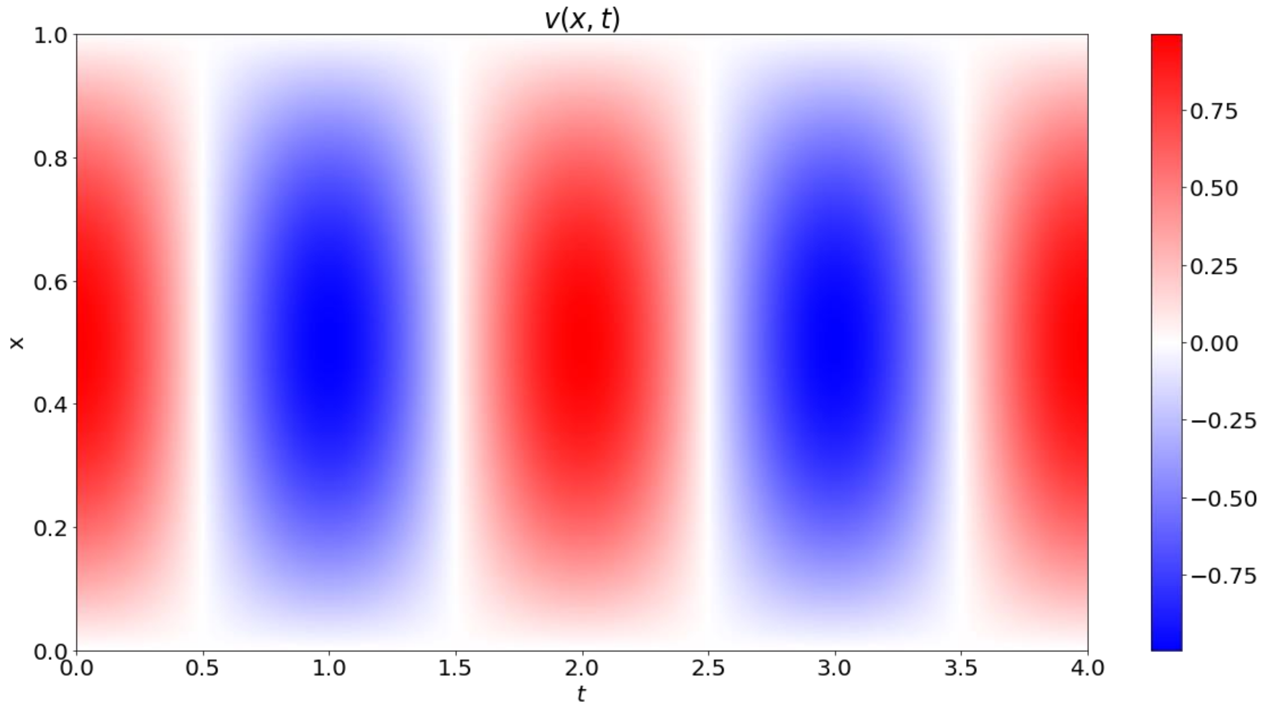
Analysis and Discussion

If we use
$f(x) = -\sin(\pi x)$
instead,
the initial attachment point shifts to the right.



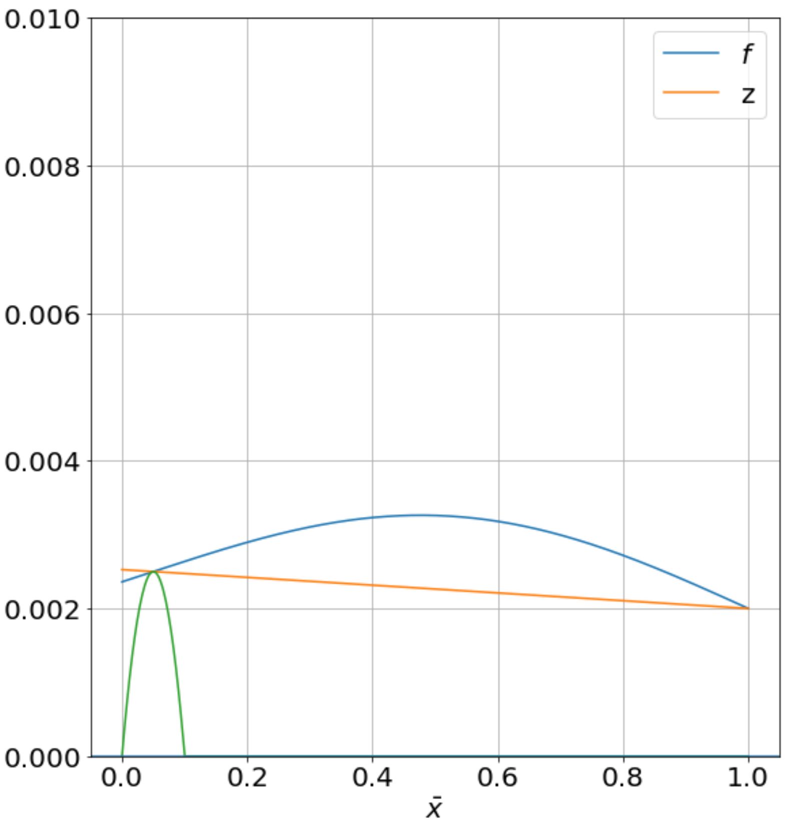
Analysis and Discussion

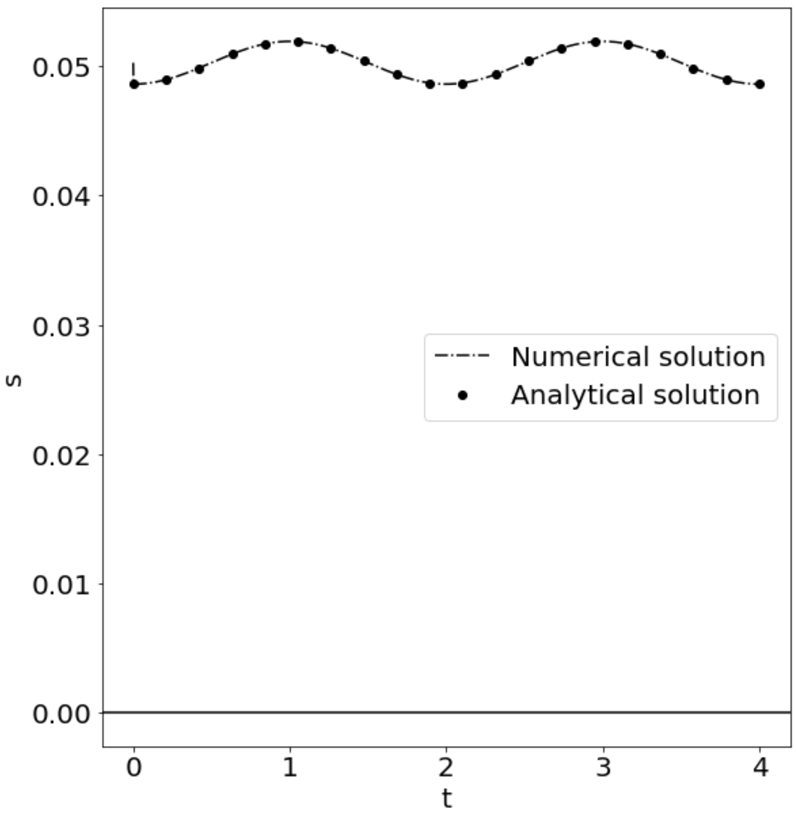
If we look at the map of v, it exactly looks like the common wave dynamics.



Analysis and Discussion

In case of smaller value of ε, i.e. 0.001, the solution still yields good result, without computational underflow





Conclusion

Conclusion

Stefan problem for wave equation has been studied

Straightforward perturbation expansion provides approximate analytical solution for the problem

Numerical schema is successfully implemented and gives close results with the analytical solution

More generic Stefan problem for wave equation is suggested for future research

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Thank you